Journal homepage www.jzs.univsul.edu.iq
Journal of Zankoy Sulaimani
Part-A- (Pure and Applied Sciences)

Results on the Domination Polynomial of Some Coalescence Graphs

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Article info
Original: 25 January 2016
Revised: 26 Octobre 2016
Accepted: 20 November 2016

Abstract

In this paper, we obtain the domination polynomial of some coalescence of a complete K_n and wheel W_n graphs such as edge introducing, vertex identification, coneighboracation graph, edge identified graph, t-tuple coalescence graphs and b-bridge tuple coalescence graphs with some special graphs.

Key Words:
T-tuple coalescence,
B-bridge coalescence,
Domination polynomial.

Introduction

Let G = (V, E) be a simple graph. The domination polynomial of a graph G of order n is the polynomial D(G; x) $=\sum_{i=\gamma(G)}^{n} d(G,i)x^i$, where d(G, i) is the number of dominating sets of G of size i, and γ(G) is the domination number of G. For any vertex u ∈ V, the open neighborhood of u is the set N(u) = {v ∈ V |uv ∈ E} and the closed neighborhood of u is the set N[u] = N(u) ∪ {u}. For a set S ⊆ V, the open neighborhood of S is N(S) = $\cup_{u\in S}$N(u) and the closed neighborhood of S is N[S] = N(S) ∪ S. A set S ⊆ V is a dominating set of G, if N[S] = V, or equivalently, every vertex in V-S is adjacent to at least one vertex in S. For more information and motivation of domination polynomials refer to [2] and [3].

Let G_1 and G_2 be two disjoint graphs. Then the vertex identification graph $G_1 \bullet G_2$ is obtained from G_1 and G_2 by identifying the vertex $u \in G_1$ with $v \in G_2$. The edge introduced graph $G_1 : G_2$ of two graphs is obtained from G_1 and G_2 by introducing the edge e=uv between a vertex $u \in G_1$ with a vertex $v \in G_2$. Let {(G_1, u_1), (G_2, u_2), …, (G_t, u_t)} be a family of not necessary distinct connected graphs with roots u_1, u_2, …, u_t, respectively. A connected graph G= $G_1 \circ G_2 \circ \dots \circ G_t$ is called the multiple coalescences of G_1, G_2,…,G_t provided that the vertices u_1, u_2, …, u_t are identified to reform the coalescence vertex u. The t-tuple coalescence graph is denoted by $G^{|t|}$ is the multiple coalescence of t isomorphic copies of a graph G. The complement G^c of G is defined by taking $V(G^c)=V(G)$ and making two vertices u and v adjacent in G^c, if and only if they are nonadjacent in G. See [4] and [5].

Domination Polynomial of Graphs

In the following, we study the definition and some results about the domination polynomial of graphs.

Definition 1: [1] and [3] Let $D(G, i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G, i) = |D(G, i)|$. Then the domination polynomial $D(G; x)$ of G is defined as
$D(G; x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i$, where $\gamma(G)$ is the domination number of G.

Theorem 1: [3] If a graph G consists of n components $G_1, \ldots, G_n$, then

$$D(G; x) = D(G_1; x) \cdot \cdot \cdot D(G_n; x).$$

Theorem 2: [2] and [3]

i) Let P_{n+1} be a path graph of order $n \geq 3$, then,
$D(P_{n+1}; x) = x[D(P_n; x) + D(P_{n-1}; x) + D(P_{n-2}; x)]$,
where $D(P_0; x) = 1$, $D(P_1; x) = x$, $D(P_2; x) = x^2 + 2x$ and $D(P_3; x) = x^3 + 3x^2 + x$.

ii) Let C_{n+1} be a cycle graph of order $n \geq 3$,, then,
$D(C_{n+1}; x) = x[D(C_n; x) + D(C_{n-1}; x) + D(C_{n-2}; x)]$,
where $D(C_0; x) = 1$, $D(C_1; x) = x$, $D(C_2; x) = x^2 + 2x$ and $D(C_3; x) = x^3 + 3x^2 + 3x$.

iii) Let K_n be a complete graph of order n, then, $D(K_n; x) = (x + 1)^n - 1$, for every $n \in N$.

iv) Let $K_{n1,n2}$ be a complete bipartite graph, then, $D(K_{n1,n2}; x) = ((x + 1)^{n1} - 1)((x + 1)^{n2} - 1) + x^{n1} + x^{n2}$.

Theorem 3: [6] and [7]

i) Let S_n be a star graph with order $n \geq 3$, then, $D(S_n; x)= \sum_{i=1}^{n}\binom{n-1}{i-1}x^i + x^{n-1}$.

ii) Let W_n be a wheel graph with order $n \geq 4$, then, $D(W_n; x)= D(S_n; x)+D(C_{n-1}; x)-x^{n-1}$.

Theorem 4: [8] Let G be a graph. For any vertex u in G we have
$D(G; x) = xD(G/u; x) +D(G - u; x) + xD(G - N[u]; x) - (x + 1)p_u(G; x)$, where $p_u(G; x)$ is the polynomial counting the dominating sets of $G - u$ which do not contain any vertex of $N(u)$ in G.

Let $G - u$ denotes the graph obtained from G by removal of u and all edges incident to u, G/u denotes the graph obtained from G by the removal of u and the addition of edges between any pair of non-adjacent neighbors of u and $G - N[u]$ (or $G - N_G[u]$) be a graph obtained by deleting all of the vertices in the closed neighborhood of u and the edges incident to them.

Corollary 1: [3] Let K^c_n be the empty graph with n vertices. Then $D(K^c_n; x) = x^n$.

Theorem 5: [1] Let $G_1= (V(G_1), E(G_1))$ and $G_2=(V(G_2), E(G_2))$ be nonempty graphs of order n_1 and n_2, respectively. Then $D(G_1\circ G_2; x)=(x(x+1)^{n2}+D(G_2; x))^{n1}$, $G_1\circ G_2$ is the corona of two graphs.

Domination Polynomial of Some Graph Coalescence

In the following, we obtain the domination polynomial of some operation coalescence of a complete graph K_n and wheel graph W_n with some special graphs.

Proposition 1: The domination polynomial of vertex identification of two complete graphs is given by
$D(K_{n1}\bullet K_{n2}; x)=xD(K_{n1+n2-2}; x)+D(K_{n1-1}; x) D(K_{n2-1};x) + x$. And $\gamma(K_{n1}\bullet K_{n2})=1$.

Proof: If $n_1=n_2=1$, then $D(K_1\bullet K_1; x)=D(K_1; x)=x$.
If $n_1=n_2=2$, then $D(K_2\bullet K_2; x)=D(P_3; x)=x^3+3x^2+x$.
If $n_1, n_2 >2$, apply Theorem 4, we have:
$D(K_{n1}\bullet K_{n2}; x)=xD(K_{n1}\bullet K_{n2}/u; x)+D(K_{n1}\bullet K_{n2}-u; x)+xD(K_{n1}\bullet K_{n2}-N[u]; x)-(x+1)p_u(K_{n1}\bullet K_{n2}; x)$,
Note that, $D(K_{n1}\bullet K_{n2}-u; x)=D(K_{n1-1}; x)D(K_{n2-1};x)$, $D(K_{n1}\bullet K_{n2}-u; x)=D(G_0; x)=1$ and $p_u(K_{n1}\bullet K_{n2}; x)=0$, then, by the observations, the result hold.
Moreover, If $n_1=n_2=n$, then $D(K_n\bullet K_n; x)=x(x+1)^{2n-2} +((x+1)^{n-1}-1)^2$.

Proposition 2: The domination polynomial of edge introducing between two complete graphs is given by
$D(K_{n1}:K_{n2}; x)=x^2D(K_{n1+n2-2}; x) + x D(K_{n1-1}; x) D(K_{n2-1}; x)+ D(K_{n1}-_1; x) D(K_{n2}; x)+ D(K_{n2-1}; x)+x^2$, for all n_1, $n_2 > 2$. And $\gamma(K_{n1}:K_{n2})=2$.

Proof: If $n_1=n_2=1$, then $D(K_1:K_1; x)=D(P_2; x)= x^2+2x$.
If $n_1=n_2=2$, then $D(K_2:K_2; x)=D(P_4; x)= x^4+4x^3+4x^2$.
If n_1, $n_2 >2$ and let $e=uv$ be an edge introducing between two complete graphs, apply Theorem 4 on vertex u, we have:
$D(K_{n1}:K_{n2}; x)= x D(K_{n1}\bullet K_{n2}; x) + D(K_{n1}-_1; x) D(K_{n2}; x) + x D(K_{n2-1}; x)$, and by Proposition 1, we get the result.

Definition 2: [5] Let G_1 and G_2 be two vertex-disjoint non empty graphs. Let $e_1=u_1 v_1 \in G_1$ and $e_2=u_2 v_2 \in G_2$, then the edge identified graph $G_1 @ G_2$ ($G_1@^{/}G_2$) of G_1 and G_2 is the graph obtained from G_1 and G_2 by identifying the vertices $u_1\equiv u_2$ and $v_1\equiv v_2$ to obtain identified vertices u and v with exactly one edge uv (respectively, $u_1\equiv v_2$ and $v_1\equiv u_2$ to obtain an identified vertices $u^{/}$ and $v^{/}$ with exactly one edge $u^{/}v^{/}$).
Proposition 3: The domination polynomial of edge identification of complete graph K_n is given by
$D(K_{n1}@K_{n2}; x)=xD(K_{n1+n2-3}; x)+xD(K_{n1+n2-4}; x)+ D(K_{n1-2}; x)D(K_{n2-2}; x)+2x$. For all $n_1, n_2 \geq 2$.
Proof: If $n_1=n_2=2$, then $D(K_2@K_2; x)=D(K_2; x)=x^2+2x$. And $D(K_2@^n; x)= D(K_2; x)$.
If $n_1, n_2> 2$, then apply Theorem 4, we have:
$D(K_{n1}@K_{n2}; x)=xD(K_{n1+n2-3}; x)+ D(K_{n1-1}\bullet K_{n2-1}; x)+x$, and by Proposition 1 , we get the result.
Moreover, If $n_1=n_2=n$, then $D(K_n@K_n; x)=x(x+1)^{2n-3}+x(x+1)^{2n-4}+((x+1)^{n-2}-1)^2$.
Definition 3: [5] Let $u_1\in G_1$ and $u_2\in G_2$, then, the coneighboracation graph $G_1© G_2$ of G_1 and G_2 is the graph whose vertex set is $V(G_1)\cup V(G_2)$ and $E(G_1© G_2)= E(G_1)\cup E(G_2)\cup\{xu_2\cup yu_1, x\in N_{G1}(u_1), y\in N_{G2}(u_2)\}$. It is clear that $p(G_1© G_2)=p_1+p_2$ and $q(G_1© G_2)=q_1+q_2+\delta_{G1}(u_1)+\delta_{G2}(u_2)$.
Proposition 4: The domination polynomial of coneighboracation of complete graph K_n is given by
$D(K_{n1}©K_{n2}; x)=xD(K_{n1+n2-1}; x)+xD(K_{n1+n2-2}; x)+ D(K_{n1-1}; x)D(K_{n2-1};x)$.
Proof: If $n_1=n_2=2$, then $D(K_2©K_2; x)= D(C_4; x)$, and $D(K_2©^n; x)= D(K_{n,n}; x)$.
Now to prove that if $n_1, n_2 > 2$, then, use Theorem 4 and Proposition 1, we get the result.
Theorem 6: The domination polynomial of t-tuple coalescence of complete graph K_n is given by:

$D(K^{|t|}_n ; x)=x[(x+1)^{t(n-1)}-1]+[(x+1)^{n-1}-1]^t+x$.

Proof: Apply Theorem 4, we have:
$D(K^{|t|}_n ; x)= xD(K_{t(n-1)}; x)+[D(K_{n-1}; x)]^t+xD(K_0; x)-(1+x)p_u(K^{|t|}_n ; x)$.
Note that, $p_u(K^{|t|}_n ; x)=0$, $D(K_0; x)=1$ and by Theorem 2 (iii), we get the result.

In the following, we obtain the domination polynomial of edge introducing graph between two t-tuple complete graphs and identified two t-tuple complete graphs by a path graph of order 3.
Proposition 5:
1) $D(K^{|t|}_{n1}: K^{|t|}_{n2}; x)= x^2 D(K_{t(n1+n2-2)}; x)+ x D(K_{t(n1-1)}; x) [D(K_{n2-1}; x)]^t+[D(K_{n1-1}; x)]^t[x[(x+1)^{t(n2-1)}-1]+[(x+1)^{n2-1}-1]^t+x]+x[D(K_{n2-1}; x)]^t+x^2$.
2) $D(K^{|t|}_{n1}(P_3)K^{|t|}_{n2}; x)=xD(K^{|t|}_{n1}:K^{|t|}_{n2}; x)+D(K^{|t|}_{n1}; x) D(K^{|t|}_{n2}; x)-[D(K_{t(n1-1)}; x)]^t[D(K_{t(n2-1)}; x))]^t$.
Proof:
1) Apply Theorem 4 on the rooted vertex of $K^{|t|}_{n1}$, we have:
$D(K^{|t|}_{n1}: K^{|t|}_{n2}; x)=x D(K_{t(n1-1)+1}\bullet K^{|t|}_{n2}; x)+ [D(K_{n1-1}; x)]^t D(K^{|t|}_{n2}; x)+x[D(K_{n2-1}; x)]^t$, and apply again Theorem 4 on the rooted vertex of $D(K_{t(n1-1)+1}\bullet K^{|t|}_{n2}; x)$, we have:
$D(K^{|t|}_{n1}: K^{|t|}_{n2}; x)=x[x D(K_{t(n1+n2-2)}; x)+ D(K_{t(n1-1)}; x) [D(K_{n2-1}; x)]^t +x]+ [D(K_{n1-1}; x)]^t D(K^{|t|}_{n2}; x)+x[D(K_{n2-1}; x)]^t$
$=x^2 D(K_{t(n1+n2-2)}; x)+x D(K_{t(n1-1)}; x) [D(K_{n2-1}; x)]^t +x^2+[D(K_{n1-1}; x)]^t D(K^{|t|}_{n2}; x)+x[D(K_{n2-1}; x)]^t$
, and by Theorem 6, we get the result.
2) Apply Theorem 4 on the rooted vertex between $K^{|t|}_{n1}$ and $K^{|t|}_{n2}$, we get the result.
Definition 4: [5] Let (u_i, G_i) be b rooted graphs, i=1, 2, …, b, where u_i is a root of G_i. Let K_1 be the trivial graph with vertex v, then the b-bridge tuple graph $K_1^{b(G_i)}$ is a graph obtained from $K_1 \cup_{i=1}^{b} G_i$ with extra b bridges, namely vu_i, i=1, 2, …, b.
Theorem 7: The domination polynomial of b-bridge coalescence of complete graph K_n is given by
$D(K_1^{b(K_n)}; x)=x[(x(x+1)^{n-1}+(x+1)^{n-1}-1)^b]+[(x+1)^n-1]^b - [(x+1)^{n-1}-1]^b$.
Proof: Apply Theorem 4, we have:
$D(K_1^{b(K_n)}; x)= D(K_1^{b(K_n)}/u; x)+ D(K_1^{b(K_n)}-u; x)+x D(K_1^{b(K_n)}- N[u]; x) - (x+1) p_u(K_1^{b(K_n)}; x)$.
Note that, $D(K_1^{b(K_n)}/u; x)=D(K_b\circ K_{n-1}; x)$, by Theorem 5, we have:
$D(K_b\circ K_{n-1}; x), =[x(x+1)^{n-1}+ (x+1)^{n-1}-1]^b$ and $D(K_1^{b(K_n)}-u; x)= [D(K_n, x)]^b$,

$D(K_1^{b(K_n)}-N[u]; x)=[D(K_{n-1}, x)]^b$ and $p_u(K_1^{b(K_n)}; x) = [D(K_{n-1}, x)]^b$.

Then, $D(K_1^{b(K_n)}; x)=x[D(K_b\circ K_{n-1}; x)]+ [D(K_n, x)]^b+x[D(K_{n-1}, x)]^b - (x+1) [D(K_{n-1}, x)]^b$.

Hence, $D(K_1^{b(K_n)}; x)= x[(x(x+1)^{n-1}+(x+1)^{n-1} - 1)^b]+ [(x+1)^n-1]^b - [(x+1)^{n-1}-1]^b$.

Proposition 6:

1) $D(K_n@C_n; x)=xD(K_n@C_{n-1}; x)+ D(K_{n-1}\bullet P_{n-1}; x)-D(P_{n-3}; x)$.

2) $D(K_n\bullet C_n; x)=xD(K_{n+1}@C_{n-1}; x)+ D(K_{n-1}; x)\, D(P_{n-1}; x)-xD(P_{n-3}; x)$.

3) $D(K_n\copyright C_n; x)=xD(K_{n+2}@C_{n-1}; x)+ D(K_n\bullet C_n; x)-xD(P_{n-3}; x)$.

4) $D(K_n{:}C_n; x)=xD(K_n\bullet C_n; x)+ D(K_{n-1}; x)\, D(C_n; x)- D(P_{n-1}; x)$.

Proof: For all cases use Theorem 4, we get the results.

Now, we define another operation of two graphs.

Definition 5: The coneighboracation of edge introduced graph $G_1\theta G_2$ of two graphs is obtained from G_1 and G_2 by introducing the edge $e=u_1u_2$ between a vertex $u_1\in G_1$ with a vertex $u_2\in G_2$. And introducing u_1 with $N_{G2}(u_2)$ and introducing u_2 with $N_{G1}(u_1)$.

In the following proposition, we obtain the domination polynomial of coneighboracation of edge introduced graph of $K_{n1}\theta K_{n2}$ and $K_{n1}\theta C_{n2}$.

Proposition 7:

1) $D(K_{n1}\theta K_{n2}; x) = x\, D(K_{n1+n2-1}) + D(K_{n1}\bullet K_{n2}; x)+x$.

2) Let K_{n1} and C_{n2} be the complete and cycle graphs of order n_1, n_2 respectively, then,
$D(K_{n1}\theta C_{n2}) = x\, D(K_{n1+2}@C_{n2-1};x) + D(K_{n1}\bullet C_{n2}; x) + x\, D(P_{n2-3};x)$.

Proof: For two cases use Theorem 4, we get the result.

Proposition 8:

$D(G\bullet P_n; x)=x[D(G\bullet P_{n-1}; x)+ D(G\bullet P_{n-2}; x)+ D(G\bullet P_{n-3}; x)],\ n \geq 3$.

Proof: Let u be the end vertex of a path graph P_n, then, apply Theorem 4, we get the recurrence relation.

Proposition 9:

1) $D(K_{n1}\bullet S_{n2}; x)= x\, D(K_{n1+n2-2}; x)+x^{n2-1}D(K_{n1-1}; x)+x$.

2) $D(K_{n1}{:}S_{n2}; x)= x^2D(K_{n1+n2-2}; x)+x^{n2}D(K_{n1-1}; x)+D(K_{n1-1}; x)\, D(S_{n2}; x)+x^{n2} +x^2$.

3) $D(K_{n1}@S_{n2}; x)= D(K_{n1}\bullet S_{n2-2}; x)$.

4) $D(K_{n1}\copyright S_{n2}; x)= x\, D(K_{n1+n2-1}; x)+xD(K_{n1+n2-2}; x)+x^{n2-1}D(K_{n1-1}; x)$.

5) $D(K_{n1}\theta S_{n2}; x) = x\, D(K_{n1+n2-1}; x)+ xD(K_{n1+n2-2}; x)+x^{n2-1}D(K_{n1-1}; x)+2x$.

Proof: For all cases use Theorem 4, we get the results.

In the following proposition, we obtain the domination polynomial of the complement of some coalescence of complete graphs.

Proposition 10:

1) $D((K_{n1}{:}K_{n2})^c; x)= D(K_{n1,n2}-e; x)$.

2) $D((K_n\bullet K_n)^c; x)=x[((x+1)^{n1-1}-1)((x+1)^{n2-1}-1)]+x^{n1}+x^{n2}$.

3) $D((K_n@K_n)^c; x)=x^2[((x+1)^{n1-1}-1)(x+1)^{n2-1}-1))]+x^{n1}+x^{n2}$.

4) $D((K_n\copyright K_n)^c; x)= (x^2+2x)[((x+1)^{n1-1}-1)(x+1)^{n2-1}-1))+ x^{n1-1}+x^{n2-1}]$.

5) $D((K_{n1}\theta K_{n2})^c; x)= x^2[((x+1)^{n1-1}-1)(x+1)^{n2-1}-1))]+x^{n1}+x^{n2}$.

Proof: For all cases directly using the definition of complement graph and Theorem 1, we get the results.

In the following, we obtain the domination polynomial of wheel graph with coalescence of some special graphs. And the coalescence between any two graphs will be with the center vertex of wheel graphs. For $n \geq 4$, the wheel graph W_n is defined to be the graph $K_1 + C_{n-1}$.

Proposition 11:

1) $D(W_n \bullet W_n; x)= x\, D(K_{2n-2}; x)+[D(C_{n-1}; x)]^2+x$.

2) $D(W_{n1} \mathbf{:} W_{n2}; x)=x^2 D(K_{n1+n2-2}; x)+xD(K_{n2-1}; x)D(C_{n2-1}; x)+D(W_{n1}; x)D(C_{n1-1}; x)+xD(C_{n1-1}; x)+x^2$.

3) $D(W_n © W_n; x)=x^2 D(K_{n1+n2-2}; x)+xD(K_{n1-1}; x)D(K_{n2-1};x) +xD(K_{2n-2}; x)+[D(C_{n-1}; x)]^2+x^2+2x$.

4) $D(W_n \theta W_n; x)=x\, D(K_{2n-1}; x) + xD(K_{2n-2}; x)+[D(C_{n-1}; x)]^2+2x$.

5) $D(W^{|t|}{}_n; x)=xD(K_{t(n-1)}; x)+[D(C_{n-1}; x)]^t+x$.

6) $D(K_1^{b(W_n)}; x)=x[x(x+1)^{n-1}+D(C_{n-1}; x)]^b+ [D(W_n; x)]^b-[D(C_{n-1}; x)]^b$.

Proof:

1) Apply Theorem 4 on the identified vertex, say u , we have:
$W_n \bullet W_n/u=K_{2n-2}$, $W_n \bullet W_n-u=C_{n-1} \cup C_{n-1}$, $W_n \bullet W_n-N[u]=G_0$ and $p_u(W_n \bullet W_n; x)=0$, then, $D(W_n \bullet W_n; x)= xD(W_n \bullet W_n/u; x)+D(W_n \bullet W_n-u; x)+xD(W_n \bullet W_n-N[u] ; x) - (x+1)\, p_u(W_n \bullet W_n; x)= xD(K_{2n-2}; x)+D(C_{n-1} \cup C_{n-1}; x)+xD(G_0; x) - (x+1)p_u(W_n \bullet W_n; x)$,
Therefore, $D(W_n \bullet W_n; x)= x\, D(K_{2n-2}; x)+[D(C_{n-1}; x)]^2+x$.

2) Let u be the center vertex in W_{n1} and v be the center vertex in W_{n2}, the edge introducing between W_{n1} and W_{n2} is $W_{n1} \mathbf{:} W_{n2}$ and e=uv. First, apply Theorem 4 on the vertex u in W_{n1} and again apply Theorem 4 on the vertex v in W_{n2}, we get the result.

3) Use Theorem 4; we get the result.

4) Use Theorem 4; we get the result.

5) Apply Theorem 4 on the tuple coalescence vertex, say u, we get the result.

6) Apply Theorem 4; we have:
$D(K_1^{b(W_n)}; x)=xD(K_1^{b(W_n)}/u; x)+ D(K_1^{b(W_n)}-u; x) + xD(K_1^{b(W_n)}- N[u]; x) - (x+1)\, p_u(K_1^{b(W_n)}; x)$.
Note that, $D(K_1^{b(W_n)}/u; x)=D(C_b \circ C_{n-1}; x)$, by Theorem 5, we have:
$D(C_b \circ C_{n-1}; x)=[x(x+1)^{n-1}+ D(C_{n-1}; x)]^b$ and $D(K_1^{b(W_n)}-u; x)=[D(W_n; x)]^b$,
$D(K_1^{b(W_n)}-N[u]; x)=[D(C_{n-1}; x)]^b$, and $p_u(K_1^{b(W_n)}; x)= [D(C_{n-1}; x)]^b$.
Then, $D(K_1^{b(W_n)}; x)=x[x(x+1)^{n-1}+ D(C_{n-1}; x)]^b + [D(W_n; x)]^b-[D(C_{n-1}; x)]^b$.

Proposition 12:

1) $D(W_n \bullet K_n; x)=x\, D(K_{2n-2}; x)+ D(W_{n-1}; x)\, D(K_{n-1}; x)+x$.

2) $D(W_n \mathbf{:} K_n; x)=x\, D(K_n \bullet K_n; x)+ D(K_n; x)\, D(C_{n-1}; x)+x\, D(K_{n-1}; x)$.

3) $D(W_n © K_n; x)=x\, D(K_{2n-1}; x)+ D(W_n \bullet K_n; x)-x$.

4) $D(W_n @ K_n; x)=xD(K_n @ K_{n-1}; x)+D(K_n @ C_{n-1}; x)+x\, D(K_{n-2}; x)$.

5) $D(W_n \theta K_n; x)=xD(K_{2n-1}; x)+D(K_n \bullet C_{n-1}; x)+x$.

Proof: For all cases apply Theorem 4 on the center vertex of wheel graph, we get the results.

Proposition 13:

1) $D(W_n \bullet C_n; x)=x\, D(K_{2n-2}; x)+ D(C_{n-1}; x)D(P_{n-1}; x)+x\, D(P_{n-3}; x)$.

2) $D(W_n \mathbf{:} C_n; x)=x\, D(K_{n-1} \bullet C_n; x)+ D(C_n; x)\, D(C_{n-1}; x)+x\, D(P_{n-1}; x)$.

3) $D(W_n © C_n; x)=x\, D(K_n \bullet K_n; x)+ D(W_n \bullet C_n; x)+x^2$.

4) $D(W_{n+1} @ C_n; x)=x\, D(K_n @ C_n; x)+ D(C_n @ C_n; x)+ xD(P_{n-2}; x)$.

5) $D(W_n \theta C_n; x)=x\, D(K_{2n-1}; x)+ D(W_n \bullet C_n; x)+x$

Proof: For all cases apply Theorem 4 on the center vertex of wheel graph, we get the results.

Proposition 14:

1) $D(W_{n1}\bullet S_{n2}; x)=x\,D(K_{n1+n2-2}; x)+x^{n2-1}D(C_{n1-1}; x)+x.$

2) $D(W_{n1}\mathbf{:}S_{n2}; x)= x\,D(K_{n1+n2-2}; x)+x^{n2-1}D(C_{n1-1}; x)+ D(C_{n1-1}; x)D(S_{n2}; x)+x^{n2}+x.$

3) $D(W_{n1}©S_{n2}; x)= xD(K_{n1+n2-2}; x)+xD(K_{n1+n2-2}; x)+x^{n2-1}D(C_{n1-1}; x).$

4) $D(W_{n1}@S_{n2}; x)= D(W_{n1}\bullet S_{n2-2}; x).$

5) $D(W_{n1}\theta S_{n2}; x)= xD(K_{n1+n2-2}; x)+xD(K_{n1+n2-2}; x)+x^{n2-1}D(C_{n1-1}; x)+2x.$

Proof: For all cases apply Theorem 4 on the center vertex of wheel graph, we get the results.

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